数理解析研究所講究録
第2046巻 2017年 128-139


# Kato's inequality when $\Delta_p u$ is a measure and related topics

堀内 利郎 ( Toshio HORIUCHI )

劉 暁静 (Xiaojing LIU)

Ibaraki University, Mito, Ibaraki, Japan.

平成 29 年 1 月 16 日

## 目 次

# 1 Introduction

$\Omega$ : a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$.

$$\Delta u = \mathrm{div}\,(\nabla u), \quad \nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N).$$

## 1.1 Convex type inequality

**Lemma 1** *(The classical convex type Kato's inequality ) Let* $u \in L^1_{\mathrm{loc}}(\Omega)$ *s.t.* $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, *then* $\Delta|u|$ *and* $\Delta u^+$ *are Radon measures and we have*

$$\Delta|u| \geq \mathrm{sgn}(u)\Delta u \qquad \text{in } D'(\Omega), \tag{1}$$

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \qquad \text{in } D'(\Omega), \tag{2}$$

*where* $\mathrm{sgn}(s) = 1$ *if* $s > 0$, $-1$ *if* $s < 0$ *and zero at* $s = 0$ $u^+ = \max[u, 0]$.

**Remark 1.1** *1. If we assume in addition that* $u$ *is continuous in* $\Omega$, *then we have*

$$\Delta|u| = \mathrm{sgn}(u)\Delta u \qquad \text{in } D'([u \neq 0]). \tag{3}$$

*The inequality (1) ;* $\Delta|u| \geq \mathrm{sgn}(u)\Delta u \quad$ *in* $D'(\Omega)$

*is a consequence of the fact that* $|u|$ *takes its minimum on the set* $[u = 0]$.

*2. Similar inequalities hold*

*when* $\Delta u$ *is replaced by elliptic operator* $M(x, \partial_x)$*:*

$$M(x, \partial_x)u = \sum_{j.k=1}^{N} \partial_{x_j}\left(a_{j,k}(x)\partial_{x_k}u\right),$$

*where* $a_{j,k}(x) \in C^1$, *and for some* $C > 0$

$$\sum_{j.k=1} a_{j,k}(x)\xi_j\xi_k \geq C|\xi|^2, \quad \text{for any } \xi \in \mathbf{R}^N$$

## 1.2 Concave type inequality

**Definition 1** (Truncation) *:* $T_k(s)$ : *Given* $k > 0$, *we denote by* $T_k$*:*$\mathbf{R} \to \mathbf{R}$ *a truncation function*

$$T_k(s) := \begin{cases} k & if \ s \geq k, \\ s & if \ -k < s < k, \\ -k & if \ \ s \leq -k. \end{cases} \tag{4}$$

Since $T_k|_{\mathbf{R}_+}$ is concave, we have the following lemma:

**Lemma 2** *Assume that $u \in L^1_{\rm loc}(\Omega)$, $\Delta u \in L^1_{\rm loc}(\Omega)$ and $u \geq 0$ a.e. in $\Omega$. Then, for any $k \geq 0$ we have*

$$\Delta(T_k(u)) \leq \chi_{[0\leq u\leq k]}\Delta u \quad \text{in } D'(\Omega), \tag{5}$$

*where $\chi_S(x)$ is a characteristic function of $S \subset \mathbf{R}$.*

Moreover, when$\Delta u$ can be replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in L^1_{\rm loc}(\Omega)$.
Here, $p$-Laplace operator is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

**Example 1** *(Classical)*
*Let $1 < p < \infty$. For $u \in K_p(\Omega)$ we have*

1. *(Convex type):*

$$\Delta_p|u| \geq \operatorname{sgn}(u)\Delta_p u \qquad \text{in } D'(\Omega), \tag{6}$$

$$\Delta_p u^+ \geq \chi_{[u\geq 0]}\Delta_p u \qquad \text{in } D'(\Omega). \tag{7}$$

2. *(Concave type): If $u \geq 0$, the we have*

$$\Delta_p T_k(u) \leq \chi_{[0\leq u\leq k]}\Delta_p u \qquad \text{in } D'(\Omega). \tag{8}$$

*Here $K_p(\Omega)$ is given by*

$$K_p(\Omega) = \{u \in L^1_{loc}(\Omega) : \partial_j u, \partial^2_{j,k} u \in L^{p^*}_{loc}(\Omega),$$
$$|\nabla u|^{p-2}|\partial^2_{j,k}u| \in L^1_{loc}(\Omega) \text{ for } j,k = 1,2,...,N\},$$

*where $p^* = \max[(p-1), 1]$.*

## 2 Main Aim

Consider a class of second order elliptic operators $\mathcal{A}$ including $\Delta_p$ and establish improved Kato's inequalities when $\mathcal{A}u$ is a Radon measure.

$$\mathcal{A}u = \operatorname{div} A(x, \nabla u), \tag{9}$$

where $A : \Omega \times R^N \mapsto R^N$ satisfies the following assumptions for some positive numbers $c_1, c_2$ and $c_3$:

1. the function $x \mapsto A(x,\xi)$ is bounded measurable for ${}^\forall\xi \in R^N$,

2. the function $\xi \mapsto A(x,\xi)$ is continuous for a.e. $x \in \Omega$,

3. $$|A(x,\xi) - A(x,\eta)| \le c_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \quad {}^\forall\xi,\eta \in R^N, \text{ a.e. } x \in \Omega,$$

4. $$(A(x,\xi) - A(x,\eta)) \cdot (\xi - \eta) \ge c_3(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2, \ {}^\forall\xi,\eta \in R^N, \text{ a.e. } x \in \Omega,$$

5. $$A(x,\lambda\xi) = \lambda|\lambda|^{p-2}A(x,\xi), \quad \text{for all } \lambda \in R, \lambda \neq 0.$$

**Remark 2.1** *1. It follows from the assumption 4 that we have*

$$A(x,\xi) \cdot \xi \ge c_1|\xi|^p \quad \textit{for all } \xi \in R^N \textit{ and a.e. } x \in \Omega.$$

*2. For some $C > 0$*

$$\sum_{j,k=1}^{N} \left| \frac{\partial A_j}{\partial \xi_k}(x,\xi) \right| \le C|\xi|^{p-2}, \quad {}^\forall\xi \in R^N \setminus \{0\}, \ \textit{a.e. } x \in \Omega, \tag{10}$$

*Then $A$ satisfies the assumptions 3 and 4.*

**Example 2** *1. In the case of $\Delta_p$, $A = A(\xi) = |\xi|^{p-2}\xi$, and $A$ satisfies the estimate (10).*

*2. Assume that $a_{j,k} \in L^\infty(\Omega)$, $a_{j,k} = a_{k,j}$ for $j,k = 1,2,\ldots,N$ and $\{a_{j,k}\}$ satisfies the uniformly elliptic estimate:*

$$\sum_{j,k=1}^{N} a_{j,k}\xi_j\xi_k \ge C|\xi|^2 \quad \textit{for any} \quad \xi \in R^N.$$

$$\mathcal{B}u = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)|\nabla u|^{p-2}\frac{\partial u}{\partial x_k}\right). \tag{11}$$

*If $p$ is sufficiently close to 2, then the operator $\mathcal{B}$ satisfies the assumptions $1 \sim 5$ with $A_j(x,\xi) = \sum_{k=1}^{N}\left(a_{j,k}(x)|\xi|^{p-2}\xi_k\right)$.*

**Definition 2** *( $M(\Omega)$: the space of Radon measure):*

*$\mu \in M(\Omega) \iff$ For every open set $\omega \subset\subset \Omega$, ${}^\exists C_\omega > 0$ s.t. $|\int_\Omega \varphi\, d\mu| \le C_\omega||\varphi||_{L^\infty}$, for ${}^\forall\varphi \in C_0^\infty(\omega)$.*

*We do not assume the finiteness of the total measure $|\mu|(\Omega) < \infty$ but assume $|\mu|(\omega) < \infty$ for each $\omega \subset\subset \Omega$.*

## 3 Decomposition of Radon measures

For any $\mu \in M(\Omega)$, $\mu$ can be uniquely decomposed as a sum of two Radon measures on $\Omega$ (see e.g. [7, 10]) : $\mu = \mu_d + \mu_c$, where

$$\begin{cases} \mu_d(A) = 0 & \text{for any Borel set } A \subset \Omega \text{ s.t } C_p(A, \Omega) = 0, \\ \\ |\mu_c|(\Omega \setminus F) = 0 & \text{for some Borel set } F \subset \Omega \text{ s.t } C_p(F, \Omega) = 0. \end{cases}$$

Total measure: $|\mu| = \mu^+ + \mu^-$.

**Definition 3** *(A p-capacity relative to $\Omega$)*
*For each compact set $K \subset \Omega$,*

$$C_p(K, \Omega) = \inf\{\int_\Omega |\nabla\varphi|^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \textit{ in some nbd of } K\}.$$

Note that $(\mu_d)^+ = (\mu^+)_d$ and $(\mu_c)^+ = (\mu^+)_c$ by the definition.

## 4 Definition of admissible class

**Definition 4 (Admissible class in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$)**
*Let $p^* = \max(1, p-1)$.*

*A function $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is said to be admissible iff $\mathcal{A}u \in M(\Omega)$ and there exists a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ s.t:*

1. *$u_n \to u$ a.e. in $\Omega$, $u_n \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $n \to \infty$.*

2. *$\mathcal{A}u_n \in L^1_{\mathrm{loc}}(\Omega)$ $(n = 1, 2, \cdots)$ and*

$$\sup_n |\mathcal{A}u_n|(\omega) = \sup_n \int_\omega |\mathcal{A}u_n| < \infty \qquad \textit{for every } \omega \subset\subset \Omega. \tag{12}$$

## 5 Some results on the admissibility

1. If $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is admissible $\Longrightarrow u^+ = \max[u, 0]$, $\quad u^- = \max[-u, 0]$, $\quad T_k(u)$ are admissible.

2. $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for $^\forall k > 0$. Moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, $^\exists C > 0$ independent on $u$ s.t

$$\begin{cases} \int_\omega |\nabla T_k(u)|^2 \leq k\left(\int_{\omega'} |\Delta u| + C\int_{\omega'} |u|\right), & \text{if } p = 2, \\ \int_\omega |\nabla T_k(u)|^p \leq Ck\left(\int_{\omega'} |\Delta_p u| + \int_{\omega'} |\nabla u|^{p-1}\right) & \text{if } p \neq 2, \end{cases}$$

3. When $p = 2$ and $\mathcal{A} = \Delta$,
$u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $\Delta u \in M(\Omega) \Longrightarrow$ u is admissible.

4. $u \in W^{1,p}_0(\Omega)$, $\mathcal{A}u \in M(\Omega) \Longrightarrow$ u is admissible.

# 6 Counter-example due to J.Serrin

Let $\Omega$ be a unit ball $B_1 = \{x \in R^N : |x| < 1\}$, and set

$$a_{i,j} = \delta_{i,j} + (a-1)\frac{x_i x_j}{r^2}, \qquad (r = |x|), \tag{13}$$

$$\mathcal{B}u = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial U}{\partial x_k}\right) = 0. \tag{14}$$

Then we have a pathological weak solution of the form

$$U(x) = x_1 r^{-\alpha}, \qquad \text{where } \alpha = \frac{N}{2} + \sqrt{\left(\frac{N}{2}-1\right)^2 + \frac{N-1}{a}}. \tag{15}$$

If $a > 1 \Longrightarrow N-1 < \alpha < N$.

**Proposition 1** *Assume that $a > 1$. Then $U \in W^{1,1}_{\mathrm{loc}}(B_1)$ and $\mathcal{B}U = 0$ in $D'(B_1)$. But $U$ is not admissible, and $\mathcal{B}(U^+)$ is not a Radon measure.*

# 7 Main results and Applications

In the rest of this note, we assume for the sake of simplicity

$$\mathcal{A} = \Delta_p.$$

## 7.1 Improved Concave type inequality

**Theorem 1** *[15, 16]*
*Assume that $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ and $u$ is addmissible.*

$$\Downarrow\Downarrow\Downarrow$$

*If $u \geq 0$ a.e. in $\Omega$, then $\Delta_p(T_k(u))$ is a Radon measure for every $k > 0$. Moreover, we have*

$$\Delta_p(T_k(u)) \leq (\Delta_p u)^+. \tag{16}$$

## 7.2 Application to Strong Maximum Principle

**Theorem 2** *[15] Let $\Omega$ be a bounded domain of $\mathbf{R}^N$. Assume that $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$, $u \geq 0$ a.e. and $u$ is admissible. Then*

1. *There exists a quasicontinuous function ( w.r.t. $C_p$ ) $\tilde{u} : \Omega \mapsto \mathbf{R}$ such that $u = \tilde{u}$ a.e. in $\Omega$.*

*2. Assume that*

$$-\Delta_p u \geq 0 \text{ in } \Omega \quad \text{in the sense of measures.} \tag{17}$$

*If $\tilde{u} = 0$ on some $K \subset \Omega$ with $C_p(K, \Omega) > 0$, then $u = 0$ a.e. in $\Omega$.*

**Remark 7.1** *$-\Delta_p u$ can be replaced by $-\Delta_p u + au^q$, where $0 \leq a \in L^1_{loc}(\Omega)$ and $q \geq p-1$.*

**Example 3** *$U = x_1/|x|^\alpha$ is not admissible in $\Omega = B_1$. Moreover $U = 0$ on $\{x_1 = 0\} \cap B_{1/2}$ which has positive p-capacity.*

### 7.3 A quick sketch of the proof of Theorem 2

$$\text{Since} \quad \Delta_p u \leq 0 \text{ in } \Omega \quad \text{in the measure sense,}$$

$$\Downarrow$$

$$(\Delta_p u)_d^+ = 0$$

$$\Downarrow$$

$$\text{Since} \quad T_k(u) \in W^{1,p}_{\text{loc}}(\Omega), \quad \Delta_p(T_k(u)) \in M(\Omega) \quad \text{for any } k > 0,$$

$$\Delta_p(T_k(u)) \leq (\Delta_p u)_d^+ = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \forall k > 0.$$

$$\Downarrow$$

$$\text{Now we can assume that} \quad u \in L^\infty(\Omega)$$

$$\Downarrow$$

As a test function, using $\varphi_0^p/(u+\delta)^{p-1}$ with $\varphi_0 = 1$ on $\omega$,

$$\int_\omega \left|\nabla \log\left(\frac{u}{\delta} + 1\right)\right|^p \leq C \int_\Omega (\varphi_0^p + |\nabla\varphi_0|^p).$$

$$\Downarrow$$

Let $E \subset \Omega$ with $C_p(E, \Omega) > 0$ s.t. $\widetilde{u} = 0$ on $E \subset \omega \subset\subset \Omega$.

By **the Poincare's inequality**

$$\int_\omega \left|\log\left(\frac{u}{\delta} + 1\right)\right|^p \leq C \int \varphi_0^p + |\nabla\varphi_0|^p \qquad \forall \delta > 0.$$

$$\Downarrow$$

We conclude that $u = 0$ a.e. in $\Omega$. □

## 7.4 Convex type Kato's inequality

**Theorem 3** *[15, 16] Let $\Phi$ be a $C^1$ convex function s.t $0 \leq \Phi' < \infty$. Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and $u$ is addmissible. Then we have*

$$\Delta_p \Phi(u) \geq \Phi'(u)^{p-1}(\Delta_p u)_d - ||\Phi'||_{L^\infty(\mathbf{R})}(\Delta_p u)_c^- \quad in \quad D'(\Omega). \tag{18}$$

**Corollary 1** *Assume the same assumptions in Theorem 3.*
*Then it holds that*

$$\Delta_p(u^+) \geq \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad in\ D'(\Omega), \tag{19}$$

$$\Delta_p|u| \geq \mathrm{sgn}(u)(\Delta_p u)_d - |\Delta_p u|_c \qquad in\ D'(\Omega), \tag{20}$$

*where* $\mathrm{sgn}(t) = 1$ *for* $t > 0$, $\mathrm{sgn}(t) = -1$ *for* $t < 0$, *and* $\mathrm{sgn}(0) = 0$.

**Example 4** *Let $u = |x|^\alpha$ for $\alpha = (p-N)/(p-1)$ and $0 \in \Omega$.*

1. *$u$ satisfies $\Delta_p u = \alpha|\alpha|^{p-2} c_N \delta$, $\delta$ : a Dirac mass, $c_N$ : the surface area of the unit ball $B_1$. If $p > 2 - 1/N$, then $|\nabla u| \in L^1_{loc}(\Omega)$ and $u$ is addimible.*

   *Recall* $\quad \Delta_p(u^+) \geq \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad inD'(\Omega). \qquad (19)$

2. *If $2 - 1/N \leq p \leq N$, then $\alpha \leq 0$, $C_p(\{0\}, \Omega) = 0$ ($\Delta_p(u^+)$ is concentrated) $(\Delta_p(u^+))_c = (\Delta_p u)_c = -(\Delta_p u)_c^- = \alpha|\alpha|^{p-2} c_N \delta \leq 0$.*

   *If $p > N$, then $\alpha > 0$, $C_p(\{0\}, \Omega) > 0$ ($\Delta_p(u^+)$ is diffuse) $(\Delta_p(u^+))_c = (\Delta_p u)_c = (\Delta_p u)_c^- = 0$ and $\Delta_p(u^+) = \chi_{[u\geq 0]}(\Delta_p u)_d = \alpha|\alpha|^{p-2} c_N \delta \geq 0$.*

   *Consequently*

$$\Delta_p(u^+) = \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad inD'(\Omega).$$

## 7.5 Inverse maximum principle

**Theorem 4** *[15, 16]* **(Inverse maximum principle )** *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$, $u \geq 0$ and $u$ is admissible. Then we have*

$$(-\Delta_p u)_c \geq 0 \quad on\ \Omega. \tag{21}$$

**Corollary 2** *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and $u$ is admissible. Then we have*

$$(-\Delta_p(u^+))_c = (-\Delta_p u)_c^+ \quad on\ \Omega. \tag{22}$$

### 7.6 A quick sketch of the proof of Theorem 4:

Recall:

$T_k(u) \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\Delta_p(T_k(u)) \in M(\Omega)$ for $\forall k > 0$.
Moreover we have

$$\Delta_p(T_k(u)) \le (\Delta_p u)^+ \quad \text{in} \quad D'(\Omega).$$

Set $\Delta_p u = \mu \in M(\Omega)$.
For some compact set $K$, s.t. $|\mu_c|(\Omega \setminus K) = 0$, $C_p(K, \Omega) = 0$.

*Then* $\quad \Delta_p T_k(u) \le \mu^+ \quad$ in $D'(\Omega \setminus K)$.

As $k \to \infty$, $\quad \mu = \Delta_p u \le \chi_{\Omega\setminus K}\mu^+ \quad$ in $D'(\Omega)$.

*Then* $\quad \mu_c|_K = \mu_K \le 0 \quad$ in $D'(\Omega)$.

$$\Downarrow$$

$$\mu_c \le 0 \quad \text{in } D'(\Omega). \tag{23}$$

□

### 7.7 Application of IMP

**Theorem 5** *[17]Suppose that $u$ is admissible. Then* $\operatorname{supp} \mu_c^{\pm} \subset \{x : u = \pm\infty\}$ *for* $\mu = \Delta_p u$.

**Remark 7.2** *From this fact,*
*if $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is an admissible solution of of $-\Delta_p u = \mu \in M(\Omega)$,*
*then $u$ is also a (local) renormalized solution of $-\Delta_p u = \mu$.*

### 7.8 A quick sketch of proof of Theorem 5

Suppose that $u$ is admissible.

$$\Downarrow$$

$$\Delta_p u = \Delta_p(T_k u) + \Delta_p(u-k)^+ - \Delta_p(u+k)^-$$

$$-\Delta_p \mu = \mu_d + \mu_c^+ - \mu_c^-$$

$$\Downarrow$$

$$\Delta_p(u-k)^+, \Delta_p(u+k)^- \le 0 \qquad (IMP)$$

$$\Downarrow$$

Note that $\Delta_p(T_k u)$ is diffuse and $k$ is an arbitrary number.

$$\Downarrow$$

$$\operatorname{supp} \mu_c^{\pm} \subset \{x : u = \pm\infty\}$$

□

# 8 Existence of admissible solution

**Theorem 6** *[17]Assumethat $\mu \in M(\Omega)$ and $|\mu|(\Omega) < \infty$. Then*

$$\begin{cases} -\Delta_p u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \Omega. \end{cases} \tag{24}$$

*has an admissible solution in $W_0^{1,p^*}(\Omega)$.*

The proof relies on the following lemma.

**Lemma 3** *Let $\{\mu_n\}$ satisfy $\sup_n |\mu_n|(\Omega) < \infty$ and $\{u_n\}$ be admissible. Assume that*

$$\begin{cases} -\Delta_p u_n = \mu_n, & \text{in } \Omega, \\ u_n = 0, & \text{on } \Omega. \end{cases} \tag{25}$$

*holds for $n = 1, 2, \ldots$.*

*Then , up to a subsequence, $u_n \to {}^{\exists}u \in W_0^{1,p^*}(\Omega)$ s.t. $u$ is admissible and satisfy (24) for ${}^{\exists}\mu$.*

# 9 Problems

1. (Nonlinear version of Good measure problem)
   Let $g(s)$ be continuous, nonnegative and nondecreasing on $[0, \infty)$. When does the next equation have an admissible solution? $-\Delta_p u + g(u) = \mu, \quad u|_{\partial\Omega} = 0$

2. (Nonlinear version of boundary Kato's inequality)
   If $u, \Delta_p u \in L^1$, then $\Delta_p u^+$ is a finite measure?

   Ex: Even if $p = 2$, there is a $u \in H^1(\Omega)$, s.t. $\Delta u = 0$, but $\int_\Omega |\Delta u^+| = \infty$

## 参考文献


[1] A. Ancona, Une propriété d'invariance des ensembles absorbants par perturbation d'un opérateur elliptique, Comm. PDE 4, (1979), 321-337.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez, An $L^1$- theory of existence and uniqueness of solutions of nonlinear elliptic equations, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze $4^e$ série, tome 22, No.2, (1995), 241-273.

[3] P. Bénilan, H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, J. Evol. Equ. 3, (2004), 673-770.

[4] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure deta, J. Funct. Anal. vol. 87, 1989, 149-169

[5] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Annales de l'I.H.P. section C, tome 13 no. 5 (1996), p. 539-551.

[6] H. Brezis , A. Ponce, Remarks on the strong maximum principle, Differential Integral Equations 16 (2003), 1-12.

[7] H. Brezis, A. Ponce, Kato's inequality when $\Delta u$ is a measure, C. R. Acad. Sci. Paris, Ser. I 338 (2004),599-604.

[8] H. Brezis, M. Marcus, A. Ponce, Nonlinear elliptic equations with measures revisited, in Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, and S. Klainerman, eds.), Annals of Mathematics Studies, 163 Princeton University Press, Princeton, NJ 2007, p. 55-110.

[9] Lorenzo D' Ambrosio, Enzo Mitidieri, A priori estimates and reduction principles for quasilinear elliptic problems and applications, Adv. Differential Equations 17 (2012), no. 9- 10, 935 20131000.

[10] L. Dupaigne and A. Ponce, Singularities of positive supersolutions in elliptic PDEs, Selecta Math. (N.S.) 10, (2004), 341-358.

[11] T. Horiuchi, On the relative $p$-capacity, J. Math. Soc. Japan, vol. 43, No. 3, (1991), 605-617.

[12] T. Horiuchi, Some remarks on Kato's inequality, J. of Inequal. & Appl., vol. 6, (2001), 29-36.

[13] T. Horiuchi, Kato's Inequalities for Degenerate Quasilinear Elliptic Operators, Kyungpook Mathematical Journal 2008 Vol. 48, No. 1, 15—24

[14] T. Kato, Schrödinger operators with singular potentials.Israel J. Math.13 (1972), 135-148.

[15] X. Liu, T. Horiuchi, Remarks on the strong maximum principle involving $p$-Laplacian, Hiroshima Mathematical Journal Volume 46, No.3 (November 2016)

[16] X. Liu, Toshio Horiuchi, Remarks on Kato's inequality when $\Delta_p u$ is a measure, Mathematical Journal of Ibaraki University Volume 48, 2017, P. 45-61.

[17] X. Liu, Toshio Horiuchi, Improved Kato's inequalities for measure-valued quasilineqar elliptic operators and its application, in preparation.

[18] F. Maeda, Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data, Hiroshima Math. J., 38 (2008), 51-93.

[19] G.D. Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa, 28, (1999), 741-808457-468.

[20] E. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations, Lecture Notes in Mathematics vol. 1074, 1984, 96-139.

[21] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, App. Math. Optim 12, (1984), 191-202.

[22] M.F. Bidaut-Víron, M. Garcia-Huidobro, L. Víron, Remarks on some quasilinear equations with gradient terms and measure data, arXiv:1211.6542 [math.AP] 13 Feb (2013).

[23] Bidaut-Véron M.F., Nguyen Quoc, H. and L. Véron Quasilinear Emden-Fowler equations with absorption terms and measure data, preprint